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Schwarz–Pick estimates for holomorphic mappings from the polydisk to the unit ball

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ABSTRACT

In this paper, Schwarz–Pick estimates of arbitrary order partial derivatives for holomorphic mappings from the polydisk to the unit ball are presented. We generalize the early work on Schwarz–Pick estimates of higher order derivatives for bounded holomorphic functions on the unit disk and the polydisk.

1. Introduction

For notation, let $\mathbf{D}$ be the unit disk in $\mathbb{C}$, $\mathbf{D}_n$ be the polydisk in $\mathbb{C}^n$, $\mathbf{B}_N$ be the unit ball in $\mathbb{C}^N$. A multi-index $v = (v_1, \dots, v_n)$ consists of n nonnegative integers v_i, $1 \leqslant i \leqslant n$. The degree of a multi-index v is the sum $|v| = \sum_{i=1}^n v_i$. Denote $v! = v_1! \cdots v_n!$. Given another multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$, let $v^\alpha = (v_1^{\alpha_1}, \dots, v_n^{\alpha_n})$. For vectors $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, $|z| = (\sum_{i=1}^n |z_i|^2)^{1/2}$, $\|z\|_\infty = \max_{1 \leqslant i \leqslant n} |z_i|$, and the multi-indices can be used as exponents in a product $z^v = \prod_{i=1}^n z_i^{v_i}$. In this paper, we denote $M_{X,Y}$ be the class of all holomorphic mappings f from the bounded domains $X \subset \mathbb{C}^n$ to $Y \subset \mathbb{C}^N$. $f = (f_1, \dots, f_N)$, $f_j(z) = \sum_\alpha a_{j,\alpha} z^\alpha$ for $j = 1, \dots, N$. Denote $f(z) = \sum_\alpha a_\alpha z^\alpha$ where $a_\alpha = (a_{1,\alpha}, \dots, a_{N,\alpha})$. Especially, we denote $M_{\mathbf{D},\mathbf{D}}$ be the set of all holomorphic self-mappings of $\mathbf{D}$.

The classical Schwarz–Pick estimate for bounded holomorphic functions in the unit disk was given as follows:

$$\left|\varphi'(z)\right| \leqslant \frac{1 - |\varphi(z)|^2}{1 - |z|^2}, \quad |z| < 1,$$

where $\varphi(z) \in M_{\mathbf{D},\mathbf{D}}$.

The above inequality has been generalized to the derivatives of arbitrary order. The first time was by Ruscheweyh [16] in 1985, and later independently by others [4–6,9,10,12,13]:

$$\left|\varphi^{(m)}(z)\right| \leqslant \frac{m!(1 - |\varphi(z)|^2)}{(1 - |z|^2)^m} \left(1 + |z|\right)^{m-1}, \quad |z| < 1,$$

where $\varphi \in M_{\mathbf{D},\mathbf{D}}$. Furthermore, [4] proves the inequalities for operator valued functions and discusses in detail the optimality of the inequalities.

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In the case of several complex variables, Rudin [15] gave a first derivative estimate for the bounded holomorphic functions on the polydisk, which is really a precursor to Schwarz–Pick estimates in high dimensions. [7] generalized the result of [9] to the holomorphic functions on $\mathbf{B}_N$. Later, [8] got the Schwarz–Pick lemma for holomorphic mappings between the unit balls. Using operator theory, [3] estimated arbitrary order derivatives of functions in the Schur–Agler class on the unit ball and the polydisk of $\mathbb{C}^n$. In fact, when $n = 2$, the Schur–Agler class on the polydisk is the unit ball of $H^\infty(\mathbf{D}_2)$, however, it is no longer the case when $n > 2$. [11] gave the Schwarz–Pick estimate for arbitrary derivatives of bounded functions on polydisk of $\mathbb{C}^n$ and generalized Theorem 7 of [3] to all holomorphic mappings in $M_{\mathbf{D}_n,\mathbf{D}}$:

Theorem A. *(See [11].) Let $\varphi(z) \in M_{\mathbf{D}_n,\mathbf{D}}$. For multi-index $m = (m_1, \dots, m_n)$, such that there are k indices $m_i = 0$ and otherwise $m_j > 0$, $j \neq i$, $1 \leqslant i, j \leqslant n$,*

$$\left|\frac{\partial^{|m|}\varphi(z)}{\partial z_1^{m_1}\cdots\partial z_n^{m_n}}\right| \leqslant m!\frac{1-|\varphi(z)|^2}{(1-\|z\|_\infty^2)^{|m|}}\big(1+\|z\|_\infty\big)^{|m|-n+k}.$$

The result of [9] follows when $\varphi \in M_{\mathbf{D},\mathbf{D}}$.

It is a well-known result that there are no biholomorphic mappings between $\mathbf{D}_n$ and $\mathbf{B}_n$; however, holomorphic mappings between $\mathbf{D}_n$ and $\mathbf{B}_N$ are interesting in several complex variables. There have been many results proved about such mappings since the 1970s, see [1,2,14]. Motivated by [8], in this paper we obtain the coefficient inequality on holomorphic mappings from $\mathbf{D}_n$ to $\mathbf{B}_N$; furthermore, estimates of higher order derivatives for all holomorphic mappings from $\mathbf{D}_n$ to $\mathbf{B}_N$ are given. As a result, we generalize the early work on Schwarz–Pick estimates of higher order derivatives for bounded holomorphic functions on $\mathbf{D}$ and $\mathbf{D}_n$ [3,9,11].

2. Main lemmas

We first give some important lemmas.

Lemma 2.1. *Let $\varphi(z) \in M_{\mathbf{D}_n,\mathbf{B}_N}$ and $\varphi(z) = \sum_\alpha a_\alpha z^\alpha$. Then*

$$\sum_\alpha |a_\alpha|^2 \leqslant 1. \tag{1}$$

Proof. For any $\zeta = (\zeta_1, \dots, \zeta_n) \in \mathbf{D}_n$, then $\zeta\theta := (\zeta_1 e^{i\theta_1}, \dots, \zeta_n e^{i\theta_n}) \in \mathbf{D}_n$, where $\theta_i \in \mathbb{R}$, $i = 1, \dots, n$. By the orthogonality we have

$$\begin{aligned} 1 &> \frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}\left|\varphi\big(\zeta_1 e^{i\theta_1}, \dots, \zeta_n e^{i\theta_n}\big)\right|^2 d\theta_1\cdots d\theta_n \\ &= \sum_\alpha |a_\alpha|^2\left|\zeta^\alpha\right|^2. \end{aligned} \tag{2}$$

Let $\zeta \to \partial\mathbf{D}_n$, the desired result is concluded. $\square$

Lemma 2.2. *Let $\varphi(z) \in M_{\mathbf{D}_n,\mathbf{B}_N}$ and $\varphi(z) = \sum_\alpha a_\alpha z^\alpha$. Then*

$$\left|\langle a_\alpha, a_0\rangle\right|^2 + \big(1-|a_0|^2\big)|a_\alpha|^2 \leqslant \big(1-|a_0|^2\big)^2. \tag{3}$$

Proof. Let $\psi(z) = \frac{1}{k}\sum_{l=1}^k \varphi(e^{\frac{l}{k}2\pi i}z)$, then $\psi(z) \in M_{\mathbf{D}_n,\mathbf{B}_N}$, $\psi(0) = a_0$ and

$$\psi(z) = a_0 + \sum_{l=1}^\infty \sum_{|\alpha|=lk} a_\alpha z^\alpha.$$

Let $\mathrm{Aut}(\mathbf{B}_N)$ be the automorphic group of $\mathbf{B}_N$. Denote $\sigma_\xi(w) \in \mathrm{Aut}(\mathbf{B}_N)$ which maps $\xi \in \mathbf{B}_N$ to 0, then $\sigma_\xi(w)$ can be expressed by

$$\sigma_\xi(w) = \frac{\xi - P_\xi w - \sqrt{1-|\xi|^2}\,Q_\xi w}{1-\langle w, \xi\rangle},$$

where $P_\xi w = \frac{\langle w,\xi\rangle\xi}{\langle \xi,\xi\rangle}$, $Q_\xi w = w - P_\xi w$ and $\sigma_\xi(\xi) = 0$, $\sigma_\xi(0) = \xi$, $\sigma_\xi^{-1} = \sigma_\xi$.

Let $\vartheta(z) = \sigma_{a_0} \circ \psi(z)$. Obviously, $\vartheta(z) \in M_{\mathbf{D}_n,\mathbf{B}_N}$ and $\vartheta(0) = 0$. We have

$$\begin{aligned}
\vartheta(z) &= \frac{1}{1-\langle\psi(z),a_0\rangle}\Bigg(-\big(a_0/|a_0|^2\big)\sum_{l=1}^{\infty}\sum_{|\alpha|=lk}\langle a_\alpha,a_0\rangle z^\alpha-\sqrt{1-|a_0|^2}\sum_{l=1}^{\infty}\sum_{|\alpha|=lk}a_\alpha z^\alpha \\
&\quad+\sqrt{1-|a_0|^2}\big(a_0/|a_0|^2\big)\sum_{l=1}^{\infty}\sum_{|\alpha|=lk}\langle a_\alpha,a_0\rangle z^\alpha\Bigg) \\
&= -\frac{1}{1-\langle\psi(z),a_0\rangle}\sum_{l=1}^{\infty}\sum_{|\alpha|=lk}\left(\frac{\langle a_\alpha,a_0\rangle a_0}{1+\sqrt{1-|a_0|^2}}+\sqrt{1-|a_0|^2}a_\alpha\right)z^\alpha \\
&= -\frac{1}{1-|a_0|^2}\sum_{|\alpha|=k}\left(\frac{\langle a_\alpha,a_0\rangle a_0}{1+\sqrt{1-|a_0|^2}}+\sqrt{1-|a_0|^2}a_\alpha\right)z^\alpha+\sum_{l=2}^{\infty}\sum_{|\alpha|=lk}d_\alpha z^\alpha.
\end{aligned}$$

Let $b_\alpha=-\frac{1}{1-|a_0|^2}(\frac{\langle a_\alpha,a_0\rangle a_0}{1+\sqrt{1-|a_0|^2}}+\sqrt{1-|a_0|^2}a_\alpha)$, then from (1) of Lemma 2.1,

$$\left|\frac{\langle a_\alpha,a_0\rangle a_0}{1+\sqrt{1-|a_0|^2}}+\sqrt{1-|a_0|^2}a_\alpha\right|\leqslant 1-|a_0|^2. \tag{4}$$

On the other hand, by computation,

$$\left|\frac{\langle a_\alpha,a_0\rangle a_0}{1+\sqrt{1-|a_0|^2}}+\sqrt{1-|a_0|^2}a_\alpha\right|^2=\big|\langle a_\alpha,a_0\rangle\big|^2+\big(1-|a_0|^2\big)|a_\alpha|^2. \tag{5}$$

Then the lemma is proved by (4) and (5). □

3. Main theorems and corollaries

Now we give our main results.

Theorem 3.1. *Let $\varphi\in M_{\mathbf{D}_n,\mathbf{B}_N}$. Then for multi-index $m=(m_1,\ldots,m_n)$, such that $m_i>0$, $i=1,\ldots,n$,*

$$\big|\langle\partial^m\varphi(z),\varphi(z)\rangle\big|^2+\big(1-\big|\varphi(z)\big|^2\big)\big|\partial^m\varphi(z)\big|^2\leqslant\left(m!\frac{1-|\varphi(z)|^2}{(1-\|z\|_\infty^2)^{|m|}}\big(1+\|z\|_\infty\big)^{|m|-n}\right)^2,$$

where $\partial^m\varphi(z)=\frac{\partial^{|m|}\varphi(z)}{\partial z_1^{m_1}\cdots\partial z_n^{m_n}}$.

Proof. Let $\tau(z)\in\mathrm{Aut}(\mathbf{D}_n)$, where $\mathrm{Aut}(\mathbf{D}_n)$ is the automorphic group of $\mathbf{D}_n$. Using the representation of automorphism $\mathbf{D}_n$, we have

$$\begin{aligned}
&\tau:(z_1,\ldots,z_n)\to\big(\tau_1(z),\ldots,\tau_n(z)\big),\\
&\tau_i(z)=e^{\sqrt{-1}\theta_i}\frac{z_{p(i)}-\zeta_i}{1-\bar\zeta_i z_{p(i)}},\quad i=1,\ldots,n,
\end{aligned}$$

where $\theta_i\in\mathbb{R}$, p is permutations of $\{1,\ldots,n\}$ and $|\zeta_i|<1$ for $i=1,\ldots,n$.

Let $p=id$, $\theta_i=0$, $i=1,\ldots,n$, then

$$\tau_i(z)=\frac{z_i-\zeta_i}{1-\bar\zeta_i z_i},\quad i=1,\ldots,n.$$

For $\varphi(z)\in M_{\mathbf{D}_n,\mathbf{B}_N}$, we set

$$F(z):=\varphi\big(\tau^{-1}(z)\big)=\sum_{u=0}^{\infty}\sum_{|v|=u}c_v z^v,\quad\text{where } c_v=(c_{1,v},\ldots,c_{N,v}),\ v=(v_1,\ldots,v_n).$$

Then

$$\varphi(z)=F\big(\tau(z)\big)=\sum_{u=0}^{\infty}\sum_{|v|=u}c_v\prod_{i=1}^{n}\left(\frac{z_i-\zeta_i}{1-\bar\zeta_i z_i}\right)^{v_i}.$$

By computation, for multi-index $m = (m_1, \dots, m_n)$,

$$\frac{d^{m_i}}{dz_i^{m_i}}\left(\frac{z_i - \zeta_i}{1 - \bar{\zeta}_i z_i}\right)^{v_i}\bigg|_{z_i=\zeta_i} = \begin{cases} 1, & m_i = v_i = 0, \\ 0, & m_i < v_i \text{ or } m_i > v_i = 0, \\ \frac{(\bar{\zeta_i})^{m_i - v_i}}{(1-|\zeta_i|^2)^{m_i}} \frac{m_i!(m_i-1)!}{(m_i-v_i)!(v_i-1)!}, & m_i \geqslant v_i > 0, \end{cases}$$

so if $|m| \geqslant |v|, m_i > 0$, $i = 1, \dots, n$, we have

$$\frac{\partial^{|m|}\varphi(\zeta)}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}} = \sum_{u=0}^{|m|} \sum_{|v|=u} c_v \prod_{i=1}^{n} \frac{(\bar{\zeta_i})^{m_i - v_i}}{(1 - |\zeta_i|^2)^{m_i}} \frac{m_i!(m_i - 1)!}{(m_i - v_i)!(v_i - 1)!}$$
$$= \sum_{|v|=n, v_i > 0}^{|m|} c_v \prod_{i=1}^{n} \frac{(\bar{\zeta_i})^{m_i - v_i}}{(1 - |\zeta_i|^2)^{m_i}} \frac{m_i!(m_i - 1)!}{(m_i - v_i)!(v_i - 1)!}. \tag{6}$$

For simplicity, denote (6) by

$$\partial^m \varphi(\zeta) = \sum_{v(m)} c_v A_v,$$

where $A_v = \prod_{i=1}^{n} \frac{(\bar{\zeta_i})^{m_i - v_i}}{(1-|\zeta_i|^2)^{m_i}} \frac{m_i!(m_i-1)!}{(m_i-v_i)!(v_i-1)!}$. By Lemma 2.2, we have

$$\left|\langle c_v, c_0\rangle\right|^2 + \left(1 - |c_0|^2\right)|c_v|^2 \leqslant \left(1 - |c_0|^2\right)^2.$$

Since $c_0 = \varphi(\zeta)$, use above inequality and Schwarz inequality, to obtain

$$\begin{aligned} &\left|\langle \partial^m \varphi(\zeta), \varphi(\zeta)\rangle\right|^2 + \left(1 - \left|\varphi(\zeta)\right|^2\right)\left|\partial^m \varphi(\zeta)\right|^2 \\ &= \left|\sum_{v(m)} A_v \langle c_v, c_0\rangle\right|^2 + \left(1 - |c_0|^2\right)\left|\sum_{v(m)} c_v A_v\right|^2 \\ &\leqslant \sum_{v(m)} |A_v| \sum_{v(m)} |A_v| \left|\langle c_v, c_0\rangle\right|^2 + \left(1 - |c_0|^2\right) \sum_{v(m)} |A_v| \sum_{v(m)} |A_v||c_v|^2 \\ &= \sum_{v(m)} |A_v| \sum_{v(m)} |A_v| \left(\left|\langle c_v, c_0\rangle\right|^2 + \left(1 - |c_0|^2\right)|c_v|^2\right) \\ &\leqslant \left(1 - |c_0|^2\right)^2 \left(\sum_{v(m)} |A_v|\right)^2. \end{aligned} \tag{7}$$

On the other hand,

$$\begin{aligned} \sum_{v(m)} |A_v| &= \sum_{|v|=n, v_i>0}^{|m|} \prod_{i=1}^{n} \frac{|\zeta_i|^{m_i - v_i}}{(1 - |\zeta_i|^2)^{m_i}} \frac{m_i!(m_i - 1)!}{(m_i - v_i)!(v_i - 1)!} \\ &\leqslant m! \sum_{|v|=n, v_i>0}^{|m|} \prod_{i=1}^{n} \frac{|\zeta_i|^{m_i - v_i}}{(1 - |\zeta_i|^2)^{m_i}} \frac{(m_i - 1)!}{(m_i - v_i)!(v_i - 1)!} \\ &\leqslant m! \frac{1}{(1 - \|\zeta\|_\infty^2)^{|m|}} \sum_{|v|=n, v_i>0}^{|m|} \prod_{i=1}^{n} \frac{|\zeta_i|^{m_i - v_i}(m_i - 1)!}{(m_i - v_i)!(v_i - 1)!} \\ &= m! \frac{1}{(1 - \|\zeta\|_\infty^2)^{|m|}} \prod_{i=1}^{n} \left(1 + |\zeta_i|\right)^{m_i - 1} \leqslant m! \frac{(1 + \|\zeta\|_\infty)^{|m|-n}}{(1 - \|\zeta\|_\infty^2)^{|m|}}. \end{aligned}$$

At last, combine with (7) and replace ζ with z, we prove Theorem 3.1. $\square$

Corollary 3.2. *Let $\varphi \in M_{\mathbf{D}_n, \mathbf{B}_N}$. Then for multi-index $m = (m_1, \dots, m_n)$, such that $m_i > 0$, $i = 1, \dots, n$,*

$$\left|\partial^m \varphi(z)\right| \leqslant m! \frac{(1 - |\varphi(z)|^2)^{1/2}(1 + \|z\|_\infty)^{|m|-n}}{(1 - \|z\|_\infty^2)^{|m|}},$$

where $\partial^m \varphi(z) = \frac{\partial^{|m|}\varphi(z)}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}}$.

More generally, if we denote $0! = 1$ then we have the following results.

Theorem 3.3. *Let $\varphi \in M_{\mathbf{D}_n,\mathbf{B}_N}$. Then for multi-index $m = (m_1, \ldots, m_n)$, such that there are k indices $m_i = 0$ and otherwise $m_j > 0$, $j \neq i$, $1 \leqslant i, j \leqslant n$,*

$$\left|\left\langle \partial^m \varphi(z), \varphi(z)\right\rangle\right|^2 + \left(1 - \left|\varphi(z)\right|^2\right)\left|\partial^m \varphi(z)\right|^2 \leqslant \left(m! \frac{1 - |\varphi(z)|^2}{(1 - \|z\|_\infty^2)^{|m|}}\left(1 + \|z\|_\infty\right)^{|m|-n+k}\right)^2,$$

where $\partial^m \varphi(z) = \frac{\partial^{|m|}\varphi(z)}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}}$.

Proof. If $|m| \geqslant |v|$, $m_i \geqslant 0$; $i = 1, \ldots, n$. Without loss of generality, we assume $m_1 = 0$, $m_i > 0$, $i = 2, \ldots, n$. We have

$$\begin{aligned} \frac{\partial^{|m|}\varphi(\zeta)}{\partial z_2^{m_2} \cdots \partial z_n^{m_n}} &= \sum_{u=0}^{|m|} \sum_{|v|=u,\, v_1=0} c_v \prod_{i=2}^{n} \frac{(\bar{\zeta_i})^{m_i - v_i}}{(1 - |\zeta_i|^2)^{m_i}} \frac{m_i!(m_i-1)!}{(m_i - v_i)!(v_i - 1)!} \\ &= \sum_{|v|=n-1,\, v_1=0,\, v_i>0\,(i\neq 1)}^{|m|} c_v \prod_{i=2}^{n} \frac{(\bar{\zeta_i})^{m_i - v_i}}{(1 - |\zeta_i|^2)^{m_i}} \frac{m_i!(m_i-1)!}{(m_i - v_i)!(v_i - 1)!}. \end{aligned} \tag{8}$$

For simplicity, denote (8) by

$$\partial^m \varphi(\zeta) = \sum_{v'(m)} c_v A_v.$$

Then we have

$$\begin{aligned} \sum_{v'(m)} |A_v| &= \sum_{|v|=n-1,\, v_1=0,\, v_i>0\,(i\neq 1)}^{|m|} \prod_{i=2}^{n} \frac{|\zeta_i|^{m_i - v_i}}{(1 - |\zeta_i|^2)^{m_i}} \frac{m_i!(m_i-1)!}{(m_i - v_i)!(v_i - 1)!} \\ &\leqslant m! \frac{1}{(1 - \|\zeta\|_\infty^2)^{|m|}} \prod_{i=2}^{n} \sum_{v_i=1}^{m_i} \frac{|\zeta_i|^{m_i - v_i}(m_i - 1)!}{(m_i - v_i)!(v_i - 1)!} \\ &= m! \frac{1}{(1 - \|\zeta\|_\infty^2)^{|m|}} \prod_{i=2}^{n} \left(1 + |\zeta_i|\right)^{m_i - 1} \leqslant m! \frac{(1 + \|\zeta\|_\infty)^{|m|-n+1}}{(1 - \|\zeta\|_\infty^2)^{|m|}}. \end{aligned} \tag{9}$$

For multi-index $m = (m_1, \ldots, m_n)$, such that there are k indices $m_i = 0$. Without loss of generality, we assume $m_1 = m_2 = \cdots = m_k = 0$ and otherwise $m_j > 0$, $k < j \leqslant n$. We deduce that

$$\sum_{|v|=n-k,\, v_1=\cdots=v_k=0}^{|m|} |A_v| \leqslant m! \frac{(1 + \|\zeta\|_\infty)^{|m|-n+k}}{(1 - \|\zeta\|_\infty^2)^{|m|}}. \tag{10}$$

Then from (10) and a similar analysis in proving Theorem 3.1, the proof of Theorem 3.3 is completed. □

Corollary 3.4. *Let $\varphi \in M_{\mathbf{D}_n,\mathbf{B}_N}$. Then for multi-index $m = (m_1, \ldots, m_n)$, such that there are k indices $m_i = 0$ and otherwise $m_j > 0$, $j \neq i$, $1 \leqslant i, j \leqslant n$,*

$$\left|\partial^m \varphi(z)\right| \leqslant m! \frac{(1 - |\varphi(z)|^2)^{1/2}(1 + \|z\|_\infty)^{|m|-n+k}}{(1 - \|z\|_\infty^2)^{|m|}}, \tag{11}$$

where $\partial^m \varphi(z) = \frac{\partial^{|m|}\varphi(z)}{\partial z_1^{m_1} \cdots \partial z_n^{m_n}}$.

Remark 3.5. When $N = 1$, Theorem 3.3 gives Theorem A. When $n = N = 1$, Theorems 3.1 and 3.3 yield Theorem 1.1 of [9].

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